÷ Vector - Analysis ÷

Scalar function:– let D be subset of set of real no. R. Then a rule, denoted by f, which associate each scaler $t \in D$, a unique real no (scaler) $f(t)$ is called scaler function of scalar variable t.

Vector function:– Let D be subset of set of real no R. Then a rule, denoted by $\vec{f}$, which associate each scalar $t \in D$, a unique vector $\vec{f}(t)$ is called vector function of scalar variable t.

eg: $\vec{f}(t) = t^2\hat{i} + 2t\hat{j} + t^3\hat{k}$ is a vector function of scalar variable t.

Point function: A function whose value at any point in a region of space depends upon the position of the point, is called point function or function of position.

Scalar point function: If to each point (x,y,z) of a region D in space, there corresponds a unique scalar denoted by $f(x,y,z)$, then f is called a scalar point function and we say scalar field f has been defined in D.

eg. (i) The temperature at any pt. (x,y,z) on the earth's surface at a certain time is scalar point function.

(ii) $f(x,y,z) = x^3y + y^2 + 2xz^3$ is scalar point function.

Vector point function: If to each point (x,y,z) of a region D in space, there corresponds a unique vector denoted by $\vec{f}(x,y,z)$, then $\vec{f}$ is called a vector point function and we say vector field $\vec{f}$ has been defined in D.

eg. (i) velocity at any pt. (x,y,z) with in a moving fluid at a certain time is vector point function.

(ii) $\vec{f}(x,y,z) = x\hat{i} - 2yz^2\hat{j} + xz\hat{k}$ is a vector pt. function

Derivative of vector function.

Let $\vec{r} = \vec{f}(t)$ be vector function of a single scalar variable t. Then derivative of vector function $\vec{f}(t)$ wrt scalar t is defined as

$$\frac{d\vec{r}}{dt} = \frac{d\vec{f}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t}$$, provided the limit exists.

Remarks (i) derivative of vector function wrt scalar variable is again a vector function.

(ii) If $\frac{d\vec{r}}{dt}$ exists, then $\vec{r}$ is said to be differentiable wrt t

(iii) $\frac{d^2\vec{r}}{dt^2} = \frac{d}{dt}\left(\frac{d\vec{r}}{dt}\right)$, $\frac{d^3\vec{r}}{dt^3} = \frac{d}{dt}\left(\frac{d^2\vec{r}}{dt^2}\right)$ and so on.

Constant vector: A vector is said to be constant vector iff both its magnitude and direction are fixed and do not change.

eg. vector $\vec{f} = 3\hat{i} + 3\hat{j} - 4\hat{k}$ is a Constant vector.

$\because |\vec{f}| = \sqrt{9+9+16} = \sqrt{34} = $ Constant

and $\vec{f}$ represent the p.v. of pt $(3,3,-4)$, So direction of $\vec{f}$ is fixed

$\therefore \vec{f}$ is Constant vector.

A vector $\vec{f}$ is said to be constant vector if value of $\vec{f}$ remains same when t changes.

i.e. $\vec{f}$ is independent of scalar variable t.

Derivative of vector function in terms of Components

Let $\vec{f}(t)$ be vector function of scalar variable t.

And $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ —(1) be component form of $\vec{f}(t)$.

Diff (1) wrt 't'

$$\frac{d\vec{f}}{dt} = \frac{d}{dt}\left[f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}\right] = \frac{d}{dt}\left[f_1(t)\hat{i}\right] + \frac{d}{dt}\left[f_2(t)\hat{j}\right] + \frac{d}{dt}\left[f_3(t)\hat{k}\right]$$

$$\Rightarrow \frac{d\vec{f}}{dt} = \frac{df_1}{dt}\hat{i} + f_1\frac{d\hat{i}}{dt} + \frac{df_2}{dt}\hat{j} + f_2\frac{d\hat{j}}{dt} + \frac{df_3}{dt}\hat{k} + f_3\frac{d\hat{k}}{dt}$$

$$\Rightarrow \frac{d\vec{f}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k} \quad \left[\because \frac{d\hat{i}}{dt} = \frac{d\hat{j}}{dt} = \frac{d\hat{k}}{dt} = \vec{0} \text{ as } \hat{i}, \hat{j}, \hat{k} \text{ are constant vectors}\right]$$

Art: A vector function $\vec{f}(t)$ is constant vector iff $\frac{d\vec{f}}{dt} = \vec{0}$

Proof: First let $\vec{f}(t)$ is constant vector.

$$\therefore \vec{f}(t+\Delta t) = \vec{f}(t) \quad \text{——} \ (1)$$

Now
$$\frac{d\vec{f}}{dt} = \lim_{\Delta t \to 0} \frac{\vec{f}(t+\Delta t) - \vec{f}(t)}{\Delta t}$$
$$= \lim_{\Delta t \to 0} \frac{\vec{f}(t) - \vec{f}(t)}{\Delta t} = \vec{0} \quad [\text{by } (1)].$$

Converse: Let $\frac{d\vec{f}}{dt} = \vec{0}$

To prove: $\vec{f}(t)$ is constant vector.

Let $\vec{f}(t) = f_1(t)\hat{i} + f_2(t)\hat{j} + f_3(t)\hat{k}$ be component form of $\vec{f}(t)$

$$\because \frac{d\vec{f}}{dt} = \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k}$$

$$\Rightarrow \frac{df_1}{dt}\hat{i} + \frac{df_2}{dt}\hat{j} + \frac{df_3}{dt}\hat{k} = \vec{0} \quad \left[\because \frac{d\vec{f}}{dt} = \vec{0}\right]$$

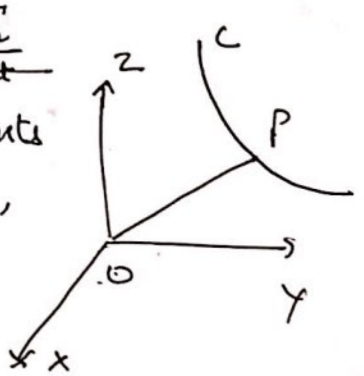
$$\Rightarrow \frac{df_1}{dt} = \frac{df_2}{dt} = \frac{df_3}{dt} = 0$$

$\Rightarrow f_1, f_2, f_3$ are independent of t

$\Rightarrow \vec{f}$ is independent of t

$\Rightarrow \vec{f}$ is constant vector function.

Geometrical Meaning of $\vec{r}$ and $\frac{d\vec{r}}{dt}$



Let O be origin and $\vec{r} = \vec{f}(t)$ represents position vector of a pt. P. As t varies, P traces out a curve C. Thus vector function $\vec{r} = \vec{f}(t)$ represents vector eqⁿ of curve in space.

Th^m: Prove that necessary and sufficient condition for $\vec{f}(t)$ to have constant magnitude is that $\vec{f} \cdot \frac{d\vec{f}}{dt} = 0$.

Proof: let $\vec{f}(t)$ has constant magnitude

Then $\vec{f} \cdot \vec{f} = |\vec{f}|^2 = \text{Constant}$

$\Rightarrow \frac{d}{dt}(\vec{f} \cdot \vec{f}) = 0 \Rightarrow \frac{d\vec{f}}{dt} \cdot \vec{f} + \vec{f} \cdot \frac{d\vec{f}}{dt} = 0$

$\Rightarrow 2\vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \Rightarrow \vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \qquad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$.

Conversely: let $\vec{f} \cdot \frac{d\vec{f}}{dt} = 0 \Rightarrow 2\vec{f} \cdot \frac{d\vec{f}}{dt} = 0$

$\Rightarrow \vec{f} \cdot \frac{d\vec{f}}{dt} + \frac{d\vec{f}}{dt} \cdot \vec{f} = 0 \qquad [\because \vec{a} \cdot \vec{b} = \vec{b} \cdot \vec{a}]$

$\Rightarrow \frac{d}{dt}(\vec{f} \cdot \vec{f}) = 0 \Rightarrow \vec{f} \cdot \vec{f} = \text{Constant}$

$\Rightarrow |\vec{f}|^2 = \text{Constant} \Rightarrow |\vec{f}| = \text{Constant}$.

Th^m: Prove that necessary and sufficient condition for $\vec{f}(t)$ to have constant direction is that $\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$.

Proof: Let $|\vec{f}(t)| = f$ and $\hat{f}$ be unit vector in the direction of $\vec{f}$. Then $\vec{f} = f\hat{f}$

$\Rightarrow \frac{d\vec{f}}{dt} = f\frac{d\hat{f}}{dt} + \frac{df}{dt}\hat{f}$

$\therefore \vec{f} \times \frac{d\vec{f}}{dt} = f\hat{f} \times \left(f\frac{d\hat{f}}{dt} + \frac{df}{dt}\hat{f}\right)$

$= f^2\left(\hat{f} \times \frac{d\hat{f}}{dt}\right) + f\frac{df}{dt}(\hat{f} \times \hat{f})$

$\Rightarrow \vec{f} \times \frac{d\vec{f}}{dt} = f^2\left(\hat{f} \times \frac{d\hat{f}}{dt}\right) \quad \text{--- (1)} \qquad [\because \hat{f} \times \hat{f} = \vec{0}]$

Let $\vec{f}(t)$ has constant direction

$\Rightarrow \hat{f}$ has constant direction, also $\hat{f}$ has constant magnitude

$\Rightarrow \hat{f}$ is a constant vector

$\Rightarrow \frac{d\hat{f}}{dt} = \vec{0}$

$\therefore$ from (1), we get $\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$.

Conversely: let $\vec{f} \times \frac{d\vec{f}}{dt} = \vec{0}$

then by ①, we get $f^2 \left(\hat{f} \times \frac{d\hat{f}}{dt} \right) = \vec{0}$

$\Rightarrow \hat{f} \times \frac{d\hat{f}}{dt} = \vec{0}$

$\Rightarrow \hat{f} \parallel \frac{d\hat{f}}{dt}$ or $\frac{d\hat{f}}{dt} = \vec{0}$ —— ②

Also $\hat{f}$ has constant magnitude. $[\because |\hat{f}| = 1]$

$\therefore \hat{f} \cdot \frac{d\hat{f}}{dt} = 0$ [A vector $\vec{f}$ has constant magnitude iff $\vec{f} \cdot \frac{d\vec{f}}{dt} = 0$]

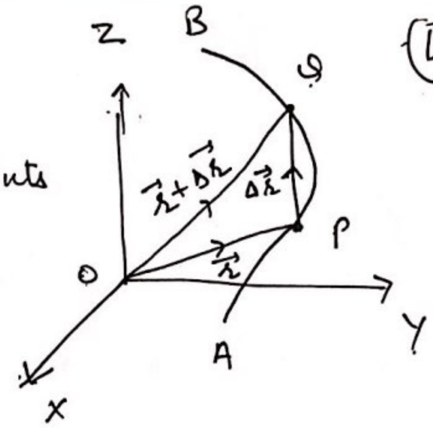
$\Rightarrow \hat{f} \perp \frac{d\hat{f}}{dt}$ or $\frac{d\hat{f}}{dt} = \vec{0}$ —— ③

Now ② and ③ holds simultaneously if $\frac{d\hat{f}}{dt} = \vec{0}$

$\Rightarrow \hat{f}$ is constant vector [$\because$ A vector $\vec{f}$ is constant vector iff $\frac{d\vec{f}}{dt} = \vec{0}$]

$\Rightarrow \hat{f}$ has constant direction.

$\Rightarrow \vec{f}$ has constant direction.



Let $\bar{r} = \bar{f}(t)$ be vector eqn of curve AB in space and $\bar{r}$, $\bar{r}+\bar{\Delta r}$ be p.v. of neighbouring points P and Q on this curve

$\therefore \ \vec{OP} = \bar{r} = \bar{f}(t)$

and $\vec{OQ} = \bar{r} + \bar{\Delta r} = \bar{f}(t+\Delta t)$

$\therefore \ \vec{PQ} = \vec{OQ} - \vec{OP} = \vec{\Delta r}$

$\Rightarrow \ \dfrac{\Delta \bar{r}}{\Delta t}$ is vector ||al to chord PQ $\left[\because \lambda\bar{a} \,||\, \bar{a}\right]$

As $Q \to P$ i.e. as $\Delta t \to 0$, chord PQ becomes tangent at P to the curve.

$\therefore \ \lim\limits_{\Delta t \to 0} \dfrac{\Delta \bar{r}}{\Delta t} = \dfrac{d\bar{r}}{dt}$ is a vector ||al to tangent at P to the curve $\bar{r} = \bar{f}(t)$.

Velocity and Acceleration of moving particle

If the scalar variable t be time and $\bar{r}$ be p.v. of moving particle P wrt origin O. Then $\bar{\Delta r}$ is the displacement of particle in time Δt and.

if velocity of particle at pt P is $\bar{v}$ then

$$\vec{v} = \lim_{\Delta t \to 0} \frac{\Delta \bar{r}}{\Delta t} = \frac{d\bar{r}}{dt}$$

and $\ \text{Acc} = \vec{a} = \dfrac{d\bar{v}}{dt} = \dfrac{d^2\bar{r}}{dt^2}$ where $\bar{a}$ is acceleration of particle at time t.

Q: Prove that the derivative of a vector point function of constant magnitude is $\perp$ar to the vector point function. Further, prove that $\left|\bar{f} \times \dfrac{d\bar{f}}{dt}\right| = |\bar{f}| \left|\dfrac{d\bar{f}}{dt}\right|$

Soln: Let $\bar{f}(t)$ be a vector pt. function s.t. $|\vec{f}(t)| = \text{Constant}$

$\Rightarrow \ |\bar{f}(t)|^2 = \text{Constant} \ \Rightarrow \ \bar{f}(t)\cdot\bar{f}(t) = \text{Constant}$

Diff both side wrt 't', we get

$\overrightarrow{f}(t) \cdot \frac{d\overrightarrow{f}(t)}{dt} + \frac{d\overrightarrow{f}(t)}{dt} \cdot \overrightarrow{f} = 0$

$\Rightarrow 2\overrightarrow{f}(t) \cdot \frac{d\overrightarrow{f}(t)}{dt} = 0$ $\quad [\because \overline{a} \cdot \overline{b} = \overline{b} \cdot \overline{a}]$

$\Rightarrow \overrightarrow{f}(t) \cdot \frac{d\overrightarrow{f}(t)}{dt} = 0 \Rightarrow \frac{d\overrightarrow{f}(t)}{dt}$ is $\perp$ar to $\overrightarrow{f}(t)$

Now $\left|\overline{f} \times \frac{d\overline{f}}{dt}\right| = |\overline{f}| \left|\frac{d\overline{f}}{dt}\right| \sin 90° \quad \left[\because \overline{f} \perp \frac{d\overline{f}}{dt}\right]$

$= |\overline{f}| \left|\frac{d\overline{f}}{dt}\right|$

Q: If $\overline{r} = x\hat{i} + y\hat{j} + z\hat{k}$ and $\overline{r} \times d\overline{r} = \overrightarrow{0}$, show that $\hat{r}$ is a constant vector.

Sol^n: Since $\overline{r} \times d\overline{r} = \overrightarrow{0}$

$\Rightarrow \overrightarrow{r}$ has constant direction. $\quad \left[\because \overline{f}(t)\right.$ has constant direction iff $\left.\overline{f} \times \frac{d\overline{f}}{dt} = \overrightarrow{0}\right]$

$\Rightarrow \hat{r}$ has constant direction

Also $\hat{r}$ is a vector of constant magnitude 1.

$\therefore \hat{r}$ is a vector having constant magnitude and direction.

$\Rightarrow \hat{r}$ is a constant vector

Q: Let $\overline{a}$ and $\overrightarrow{b}$ are constant vectors and $\overrightarrow{r} = t^m\overrightarrow{a} + t^n\overrightarrow{b}$. If $\overrightarrow{r}$ and $\frac{d^2\overrightarrow{r}}{dt^2}$ are ||al., then prove that either $m+n=1$ or $m=n$

Sol^n: Since $\overline{a}$ and $\overline{b}$ are constant vectors $\therefore \frac{d\overline{a}}{dt} = \frac{d\overrightarrow{b}}{dt} = \overrightarrow{0}$

Now $\overrightarrow{r} = t^m\overrightarrow{a} + t^n\overrightarrow{b}$ —— (1)

$\Rightarrow \frac{d\overrightarrow{r}}{dt} = mt^{m-1}\overrightarrow{a} + nt^{n-1}\overrightarrow{b}$

and $\frac{d^2\overrightarrow{r}}{dt^2} = m(m-1)t^{m-2}\overrightarrow{a} + n(n-1)t^{n-2}\overrightarrow{b}$ —— (2)

Since $\overrightarrow{r} \parallel \frac{d^2\overrightarrow{r}}{dt^2} \Rightarrow \overrightarrow{r} \times \frac{d^2\overrightarrow{r}}{dt^2} = \overrightarrow{0}$

$\Rightarrow (t^m\overrightarrow{a} + t^n\overrightarrow{b}) \times [m(m-1)t^{m-2}\overrightarrow{a} + n(n-1)t^{n-2}\overrightarrow{b}] = \overrightarrow{0}$

$\Rightarrow m(m-1)t^{2m-2}\,\overline{a} \times \overrightarrow{a} + n(n-1)t^{m+n-2}\,\overline{a} \times \overrightarrow{b}$
$+ m(m-1)t^{m+n-2}\,\overline{b} \times \overrightarrow{a} + n(n-1)t^{2n-2}\,\overline{b} \times \overline{b} = \overrightarrow{0}$

$\Rightarrow [n(n-1)t^{m+n-2} - m(m-1)t^{m+n-2}]\,\overline{a} \times \overrightarrow{b} = \overrightarrow{0} \quad \left[\because \begin{array}{l}\overline{a} \times \overline{a} = \overrightarrow{0} \\ \overline{b} \times \overrightarrow{b} = \overrightarrow{0}\end{array}\right]$

$\Rightarrow [n(n-1) - m(m-1)]\, t^{m+n-2}\, \bar{a} \times \bar{b} = \vec{0}$

$\Rightarrow n^2 - m^2 - n + m = 0 \Rightarrow (n-m)(n+m-1) = 0$

$\Rightarrow n - m = 0$ or $n + m - 1 = 0 \Rightarrow m = n$ or $m + n = 1$

Q: If $\vec{r}$ is a unit vector, prove that $\left|\bar{r} \times \frac{d\bar{r}}{dt}\right| = \left|\frac{d\bar{r}}{dt}\right|$.

Sol^n: Since $\bar{r}$ is unit vector

$\Rightarrow \bar{r}$ has constant magnitude

$\Rightarrow \bar{r} \cdot \frac{d\bar{r}}{dt} = 0$ $\left[\because \bar{f}(t) \text{ has constant magnitude iff } \bar{f} \cdot \frac{d\bar{f}}{dt} = 0\right]$

$\Rightarrow \bar{r} \perp \frac{d\bar{r}}{dt}$

Now $\left|\bar{r} \times \frac{d\bar{r}}{dt}\right| = |\bar{r}|\left|\frac{d\bar{r}}{dt}\right| \sin 90^\circ$ $\left[\because \bar{r} \perp \frac{d\bar{r}}{dt}\right]$

$= (1)\left|\frac{d\bar{r}}{dt}\right|(1)$ $[\because |\bar{r}| = 1]$

$\Rightarrow \left|\bar{r} \times \frac{d\bar{r}}{dt}\right| = \left|\frac{d\vec{r}}{dt}\right|$

Q: If $\vec{R}$ be unit vector in the direction of $\vec{r}$, prove that $\vec{R} \times \frac{d\vec{R}}{dt} = \frac{\bar{r}}{r^2} \times \frac{d\bar{r}}{dt}$, where $r = |\bar{r}|$.

Sol^n: Since $\vec{R}$ is unit vector in the direction of $\vec{r}$

$\therefore \vec{R} = \frac{\bar{r}}{r}$ —— (1)

and $\frac{d\vec{R}}{dt} = \frac{d}{dt}\left(\frac{1}{r}\bar{r}\right) = \bar{r}\, \frac{d}{dt}\left(\frac{1}{r}\right) + \frac{1}{r}\frac{d\bar{r}}{dt}$

$\Rightarrow \frac{d\vec{R}}{dt} = -\frac{1}{r^2}\frac{dr}{dt}\,\bar{r} + \frac{1}{r}\frac{d\bar{r}}{dt}$ —— (2)

$\therefore \bar{R} \times \frac{d\bar{R}}{dt} = \frac{\bar{r}}{r} \times \left(-\frac{1}{r^2}\frac{dr}{dt}\,\vec{r} + \frac{1}{r}\frac{d\bar{r}}{dt}\right)$ (by (1) and (2))

$= -\frac{1}{r^3}\frac{dr}{dt}\,\bar{r} \times \vec{r} + \frac{1}{r^2}\,\bar{r} \times \frac{d\bar{r}}{dt} = \vec{0} + \frac{1}{r^2}\,\vec{r} \times \frac{d\vec{r}}{dt}$

$\Rightarrow \vec{R} \times \frac{d\vec{R}}{dt} = \frac{1}{r^2}\,\bar{r} \times \frac{d\bar{r}}{dt}$ $[\because \bar{r} \times \bar{r} = \vec{0}]$

Q: At any point of the curve $x = 3\cos t$, $y = 3\sin t$, $z = 4t$, find. (i) tangent vector (ii) unit tangent vector (iii) normal vector (iv) unit normal vector. (7)

Soln: let $\vec{r}$ be p.v. of any pt (x, y, z) on curve

$\therefore \vec{r} = x\hat{i} + y\hat{j} + z\hat{k} = 3\cos t\,\hat{i} + 3\sin t\,\hat{j} + 4t\,\hat{k}$

(i) $\dfrac{d\vec{r}}{dt} = (-3\sin t)\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}$

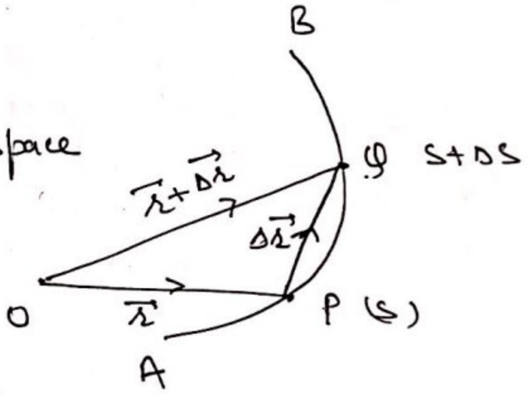
which is reqd. tangent vector.

(ii) $\left|\dfrac{d\vec{r}}{dt}\right| = \sqrt{9\sin^2 t + 9\cos^2 t + 16} = \sqrt{9+16} = \sqrt{25} = 5$

$\hat{T}$ = Unit tangent vector $= \dfrac{\frac{d\vec{r}}{dt}}{\left|\frac{d\vec{r}}{dt}\right|} = \dfrac{1}{5}\left[(-3\sin t)\hat{i} + 3\cos t\,\hat{j} + 4\hat{k}\right]$

(iii)

Let $\vec{r} = \vec{f}(s)$ be vector eqn of curve AB in space where parameter s is arc length measured from fixed pt A on curve



where $\widehat{AP} = s$, $\widehat{AQ} = s + \Delta s$

$\Rightarrow \widehat{PQ} = \Delta s$

Let $\vec{r}$ and $\vec{r} + \Delta\vec{r}$ be p.v. of two neighbouring points P and Q on this curve.

$\therefore \quad \vec{OP} = \vec{r} = \vec{f}(s)$

and $\vec{OQ} = \vec{r} + \Delta\vec{r} = \vec{f}(s + \Delta s)$

$\therefore \vec{PQ} = \vec{OQ} - \vec{OP} = \Delta\vec{r}$

$\Rightarrow \dfrac{\Delta\vec{r}}{\Delta s}$ is vector llal to chord PQ $[\because \lambda\vec{a} \parallel \vec{a}]$

As $Q \to P$ i.e. $\delta s \to 0$, chord PQ becomes tangent at P to the curve.

$\therefore \lim\limits_{\Delta s \to 0} \dfrac{\Delta\vec{r}}{\Delta s} = \dfrac{d\vec{r}}{ds}$ is vector llal to tangent at P to the curve $\vec{r} = \vec{f}(s)$

Also $\left|\frac{d\bar{r}}{ds}\right| = \lim_{\Delta s\to 0}\left|\frac{\Delta\bar{r}}{\Delta s}\right| = \lim_{Q\to P}\frac{|\Delta\bar{r}|}{PQ} = \lim_{Q\to P}\frac{\text{Chord } PQ}{\text{Arc } PQ} = 1$

$\therefore \frac{d\bar{r}}{ds}$ is unit vector along the tangent at P

and is denoted by $\hat{T}$.

Thus $\hat{T} = \frac{d\bar{r}}{ds}$ is unit tangent vector

$\Rightarrow \hat{T}$ has Constant magnitude

$\Rightarrow \hat{T}\cdot\frac{d\hat{T}}{ds} = 0$

$\Rightarrow \frac{d\hat{T}}{ds} \perp \hat{T}$

$\Rightarrow \frac{d\hat{T}}{ds}$ is vector normal at pt P to curve. Where $\hat{T}$ is unit tangent vector

(iii) Normal vector $= \frac{d\hat{T}}{ds}$ where $\hat{T}$ is unit tangent vector.

$$= \frac{d\hat{T}}{dt}\cdot\frac{dt}{ds}$$

Now $\frac{ds}{dt} = \left|\frac{ds}{dt}\right| = \left|\frac{d\bar{r}}{dt}\right| = 5$

as $\delta t \to 0$

Arc $PQ \simeq$ Chord PQ

$\Rightarrow \Delta s = |\Delta\bar{r}|$

$\Rightarrow \lim_{\Delta t\to 0}\frac{\Delta s}{\Delta t} = \lim_{\Delta t\to 0}\left|\frac{\Delta\bar{r}}{\Delta t}\right|$

$\Rightarrow \frac{ds}{dt} = \left|\frac{d\bar{r}}{dt}\right|$

$\therefore$ Normal vector $= \frac{1}{5}\frac{d\hat{T}}{dt}$

$$= \frac{1}{5}\frac{d}{dt}\left[\frac{1}{5}(-3\sin t\,\hat{i} + 3\cos t\,\hat{j} + 4\hat{k})\right]$$

$$= \frac{1}{25}(-3\cos t\,\hat{i} - 3\sin t\,\hat{j})$$

$\Rightarrow$ Normal vector $= -\frac{3}{25}\left[\cos t\,\hat{i} + \sin t\,\hat{j}\right]$

(iv) Unit normal vector $= \dfrac{-\frac{3}{25}\left[\cos t\,\hat{i} + \sin t\,\hat{j}\right]}{\left|-\frac{3}{25}(\cos t\,\hat{i} + \sin t\,\hat{j})\right|}$

$$= -(\cos t\,\hat{i} + \sin t\,\hat{j})$$

Q: If $\vec{r}\cdot d\vec{r} = 0$, Show that $r =$ Constant.

Sol^n: Let $\vec{r} = x\hat{i} + y\hat{j} + z\hat{k}$

$\therefore d\vec{r} = dx\hat{i} + dy\hat{j} + dz\hat{k}$

$\Rightarrow \vec{r}\cdot d\vec{r} = x\,dx + y\,dy + z\,dz$

$\Rightarrow x\,dx + y\,dy + z\,dz = 0 \qquad [\because \vec{r}\cdot d\vec{r} = 0]$

$\Rightarrow 2x\,dx + 2y\,dy + 2z\,dz = 0$

$\Rightarrow d(x^2+y^2+z^2) = 0$ Integrating, we get

$\Rightarrow x^2+y^2+z^2 =$ Constant $\Rightarrow r^2 =$ Constant

$\Rightarrow r =$ Constant

Q: If $\vec{R}$ is unit vector in the direction of $\vec{r}$, then show that $\vec{R}\times d\vec{R} = \dfrac{\vec{r}\times d\vec{r}}{r^2}$

Sol^n: Since $\vec{R}$ is unit vector in the direction of $\vec{r}$

$\therefore \vec{R} = \dfrac{\vec{r}}{r}$ —— (1) where $r = |\vec{r}|$

$\therefore d\vec{R} = d\left(\dfrac{\vec{r}}{r}\right) = \dfrac{r\,d\vec{r} - \vec{r}\,dr}{r^2}$ —— (2)

Now $\vec{R}\times d\vec{R} = \dfrac{\vec{r}}{r} \times \dfrac{r\,d\vec{r} - \vec{r}\,dr}{r^2}$ [by (1) and (2)]

$$= \frac{r\,\vec{r}\times d\vec{r} - \vec{r}\times\vec{r}\,dr}{r^3}$$

$$= \frac{r\,\vec{r}\times d\vec{r}}{r^3} \qquad [\because \vec{r}\times\vec{r} = \vec{0}]$$

$$= \frac{\vec{r}\times d\vec{r}}{r^2}$$
